BMO1 2013-14 REPORT FOR TEACHERS AND CANDIDATES

This year's BMO1 paper was marked over the weekend of December 6th - 8th by: Richard Atkins, Ben Barrett, Natalie Behague, Lex Betts, Ilya Chevyrev, Philip Coggins, James Cranch, Tim Cross, Paul Fannon, Mark Flanagan, Richard Freeeland, James Gazet, Ed Godfrey, Karl Hayward-Bradley, Tim Hennock, Maria Holdcroft, Ina Hughes, Ian Jackson, Andrew Jobbings, Vesna Kadelburg, Jeremy King, Gerry Leversha, Sam Maltby, David Mestel, Joseph Myers, Vicky Neale, Peter Neumann, Sylvia Neumann, Craig Newbold, David Phillips, Hannah Roberts, Dominic Rowland, Jack Shotton, Geoff Smith, Karthik Tadinada, Jerome Watson, Alison Zhu.

This year practically all scripts arrived at the UKMT Office in time for the marking weekend. We greatly appreciate your help in this matter. Please would teachers note that even clever students are not always good at reading the instructions on the paper! **So please guide them through it**. They should write on **one side of the paper only** and **start each question on a new sheet**. It also helps if they can number the questions carefully in a place which is not going to be covered by the staple!! Many thanks to all centres who managed to get their candidates to do all this as instructed.

This year's paper had a wonderful set of questions, thanks to the creativity of the BMO Setting Committee, chaired by Dr Jeremy King, and there were some excellent scripts, including three perfect scores. Over half the candidates managed to get full marks on the first question, and most candidates made serious attempts at several questions. The BMO1 Solutions Video is available for everyone to watch. It will be on-line until Jan 31st 2014 and can be found at *http://www.bmoc.maths.org/solutions/bmo1-2014/*.

There were 1526 scripts received and the mean score was 15.0. The breakdown of final marks was:

Mark	0 - 9	10 - 19	20 - 29	30 - 39	40 - 49	50 - 59	60
No. of cand.	427	634	272	118	58	13	3

The average mark per question was:

Q1	Q2	Q3	Q4	Q5	Q6
7.3	3.8	2.2	4.4	1.7	1.3

All BMO questions are marked using a "10–/0+" strategy, which requires a candidate to get the crucial part of each solution to qualify for 10- (with minor deductions for omissions/errors), otherwise they are in 0+ territory, when usually only a maximum of 3 marks are available. Where there were two parts to a question, a similar principle was used for each part.

Q1. Hopefully candidates realise that the point of BMO questions is not to make them do horrendous calculations, but to come up with creative ways to solve problems. Here the best idea was to use algebra, and the easiest substitution was to let $2013 = n$ and try to factorise. In order to factorise an expression of the form $a^4 + 4b^4$, it is a clever idea to add and subtract $4a^2b^2$, giving $(a^2 + 2b^2)^2 - 4a^2b^2$, which can be further factorised using the difference of two squares. This is sometimes called Sophie Germain's identity and is written in the form $a^4 + 4b^4 = (a^2 + (a + b)^2)(a^2 + (a - b)^2)$. In this case, conveniently $a - b = \pm 1$. Cancelling reduces both fractions to $n^2 + 1$, so the difference is zero. Alternatively, one can multiply out within each fraction and then divide, yielding the same result, though the algebra is more tedious. There were 882 perfect scores on this question.

Q2. This elegant little geometry problem needed two ideas. The first being the use of the Alternate Segment Theorem to show that the angle between AB and l was equal to angle BCE. The second idea was to note that $BECF$ was cyclic since $BEC + BFC = 180°$, and hence BCE is also equal to BFE (angles in the same segment). This completes the proof as we have two corresponding angles and hence AB is parallel to EF. An alternative proof is to let AC meet the tangent at X (if they don't meet, this is an easy special case). We still need the observation that $BECF$ is cyclic, and considering the power of the point X with respect to the circle ABC gives $XB^2 = XC . XA$ and with respect to $BECF$ gives $XF . XB = XC . XE$.

Dividing these equations gives $XB / XF = XA / XE$. Hence AB is an enlargement of EF from X, so AB and EF are parallel. Candidates do need to make sure that they don't consider only a special case (for example when l is parallel to AC). Proof for just this case was not given any marks. There were 309 perfect scores on this question.

Q3. This problem was difficult to visualise as a number with 3^{2013} digit 3s is somewhat large! Even considering small cases (3^1 threes, or 3^2 or 3^3) rapidly became unmanageable. The first idea, which must be common knowledge for all candidates, is the rule for divisibility by 3 which is possible when the digit sum is divisible by 3. Clearly the initial number is divisible by 3, and division gives a chain of 3^{2013} digit 1s. The key idea here is going to be induction, but a formal inductive proof is not required at BMO1 – an informal argument will suffice. We could divide this string of 1s by 3 to give 37037037…..037, a string of 3^{2012} groups of 037, but maybe it is better and clearer to divide by 111 (which has a **single** factor of 3), to give 1001001…001, a string of 3^{2012} groups of 001. This new number has a digit sum of 3^{2012}, so is divisible by 3. However, if we (similarly to before), divide by 1001001 (which has a **single** factor of 3), we get 1000000001000000001…000000001, a string of 3^{2011} groups of 000000001. Each time we repeat this process we get another factor of 3, giving a final divisor of 3^{2014}. A full proof must show this is attainable and maximal, and the challenge is to find a good, clear way of explaining this, which many of the candidates did. Over 100 candidates scored full marks on this question.

Q4. The most common approach with this popular question was to work through the number of options at each stage and try to derive some recurrence relation. Since what you can do each day is dependant on the last thing you did, it is helpful to have some clear notation, like r_n = the number of n day holidays ending in rest, and s_n for the number ending in sport. This would give $r_{n+1} = r_n + s_n$ and $s_{n+1} = 2r_n + s_n$. Starting with $r_1 = 1$ and $s_1 = 2$, we get $r_9 + s_9 = 3363$. 258 people scored full marks on this question.

Q5. There were several ways to approach this question. The simplest is to construct lines through P parallel to the sides of the given equilateral triangle (as shown in glorious colour on the solutions video). This produces several small equilateral triangles which are bisected by the perpendiculars from P to the sides. If we note equal side lengths into which the sides have been split, we notice that both $AF + BD + CE$ and also $AE + BF + CD$ comprise the same six lengths, and thus are equal. An alternative simple approach can be found by joining P to the vertices A, B, C and using Pythagoras' Theorem. For example, considering PD^2, we get the result that $PB^2 - PC^2 = DB^2 - DC^2 = (DB - DC)(DB + DC)$. If we do the same thing for the other two sides and note that $DB + DC = EC + CA = AF + FB =$ *side of triangle ABC*, dividing through produces the result.
The second part is again most easily done by the first construction, and noting triangles with equal area. Both sides comprise six triangles, which can be paired up in triangles of equal area. A coordinate geometry approach will also work, as will vectors, trig, polar coordinates, complex numbers, and a variational approach (moving P parallel to the sides of the triangle).
The first part was worth 4 marks, and the second part was worth 6 points in the mark scheme. Just 32 candidates scored full marks on this question.

Q6. The left-hand inequality can be proved by first noting that since the largest side is opposite the largest angle etc, so assuming without loss of generality that $a \geq b \geq c$ and therefore $A \geq B \geq C$, the inequality
$0 \leq (a-b)(A-B) + (b-c)(B-C) + (c-a)(C-A)$ gives $(A + B + C)(a + b + c) \leq 3(aA + bB + cC)$, which gives the LHS inequality.

The right-hand side follows from the triangle inequality in the form $a/(a + b + c) < ½$ and similarly for b and c. Multiplying up by A, B and C and adding these gives $\dfrac{aA+bB+cC}{a+b+c} < \dfrac{A}{2}+\dfrac{B}{2}+\dfrac{C}{2} \leq 90.$
There were 5 marks available for proving each part of the double-inequality. Well done to the 26 people who scored full marks on this question.